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Semigroup expanded algebras and gravity

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Enlargment of the algebra

Maxwell algebra [Schrader, Bacry 1977]

Schrader, *The Maxwell group and the quantum theory of particles in classical homogeneous electromagnetic fields*, Fortsch. Phys. **20** (1972) 701

Maxwell algebra, corresponds to the symmetry of fields in the constant electromagnetic background in the flat Minkowski spacetime.

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac},$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b; \qquad [P_a, P_b] = Z_{ab},$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} + \eta_{ad} Z_{bc} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac},$$

$$[Z_{ab}, P_c] = 0; \qquad [Z_{ab}, Z_{cd}] = 0.$$

Semisimple extension of Poincaré [Soroka^2]

D. V. Soroka and V. A. Soroka, "Tensor extension of the Poincaré' algebra," Phys. Lett. B **607**, 302 (2005) [hep-th/0410012]

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac},$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b; \qquad [P_a, P_b] = Z_{ab},$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} + \eta_{ad} Z_{bc} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac},$$

$$[Z_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b,$$

$$[Z_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} + \eta_{ad} Z_{bc} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac}.$$

$$A_\mu = \frac{1}{2}\omega_\mu^{ab} J_{ab} + \frac{1}{\ell} e_\mu^a P_a + \frac{1}{2} k_\mu^{ab} Z_{ab}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Cosmological constant term due to the new generator

$$\frac{1}{\ell^2} e_\mu^a e_\nu^b [P_a, P_b] \to Z_{ab}$$

J. A. de Azcarraga, K. Kamimura and J. Lukierski, "Generalized cosmological term from Maxwell symmetries," Phys. Rev. D **83**, 124036 (2011) [arXiv:1012.4402 [hep-th]].

Soroka^2 algebra as a direct sum of AdS ⊕ Lorentz

Change of basis

$$L_{IJ} = \begin{cases} L_{ab} = Z_{ab} \\ L_{a(D+1)} = P_a \end{cases}$$

$$N_{ab} = (J_{ab} - Z_{ab})$$

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac},$$
$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b; \qquad [P_a, P_b] = Z_{ab},$$
$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} + \eta_{ad} Z_{bc} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac},$$
$$[Z_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b,$$
$$[Z_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} + \eta_{ad} Z_{bc} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac}.$$

$$\{J_{ab}, P_a, Z_{ab}\}$$

$$so(D-1,2) \oplus so(D-1,1)$$

AdS $\{Z_{ab}, P_a\} \oplus$ Lorentz $\{(J-Z)_{ab}\}$

D.V. Soroka, V.A. Soroka, *Tensor extension of the Poincaré algebra*, Phys. Lett. B **607** (2005) 302 arXiv:hep-th/0410012.

D.V. Soroka, V.A. Soroka, *Semi-simple extension of the (super)Poincare algebra*, Adv. High Energy Phys. 2009 (2009) 34147 [hep-th/0605251].

$$[P_a, P_b] = J_{ab} - Z_{ab}$$

AdS $\{(J-Z)_{ab}, P_a\} \oplus$ Lorentz $\{Z_{ab}\}$

R. Durka, J. Kowalski-Glikman, M. Szczachor, *Gauged AdS-Maxwell algebra and gravity*, Mod. Phys. Lett. A **26** (2011) 2689. arXiv:1107.4728 [hep-th].

Generalized contractions --> semigroup expansion

$$[P_a, P_b] = J_{ab}$$

AdS

Inönü-Wigner contraction

$$P_a \to \ell \tilde{P}_a$$

$$[\tilde{P}_a, \tilde{P}_b] = \frac{1}{\ell^2} J_{ab} \quad \underset{\ell \to \infty}{} \quad [\tilde{P}_a, \tilde{P}_b] = 0$$

Poincaré

Generators $\{J_{ab}, P_a, Z_{ab}, Z_a\}$

$$\begin{aligned} \boldsymbol{J}_{ab} &= \lambda_0 \otimes \tilde{\boldsymbol{J}}_{ab}, \\ \boldsymbol{Z}_{ab} &= \lambda_2 \otimes \tilde{\boldsymbol{J}}_{ab}, \\ \boldsymbol{P}_a &= \lambda_1 \otimes \tilde{\boldsymbol{P}}_a, \\ \boldsymbol{Z}_a &= \lambda_3 \otimes \tilde{\boldsymbol{P}}_a. \end{aligned}$$

The semigroup elements $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ are *not* real numbers and they are *dimensionless*. In this particular case, they obey the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq 4, \\ \lambda_4, & \text{when } \alpha + \beta > 4. \end{cases}$$

$$\lambda_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \lambda_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{A} = \frac{1}{2}\omega^{ab}\boldsymbol{J}_{ab} + \frac{1}{\ell}e^a\boldsymbol{P}_a + \frac{1}{2}k^{ab}\boldsymbol{Z}_{ab} + \frac{1}{\ell}h^a\boldsymbol{Z}_a,$$

$$\boldsymbol{F} = \frac{1}{2}R^{ab}\boldsymbol{J}_{ab} + \frac{1}{\ell}T^a\boldsymbol{P}_a + \frac{1}{2}\left(\mathrm{D}_\omega k^{ab} + \frac{1}{\ell^2}e^a e^b\right)\boldsymbol{Z}_{ab} + \frac{1}{\ell}\left(\mathrm{D}_\omega h^a + k^a{}_b e^b\right)\boldsymbol{Z}_a.$$

$$\begin{aligned} \langle \boldsymbol{J}_{ab}\boldsymbol{J}_{cd}\boldsymbol{P}_e \rangle &= \frac{4}{3}\ell^3\alpha_1\varepsilon_{abcde}, \\ \langle \boldsymbol{J}_{ab}\boldsymbol{J}_{cd}\boldsymbol{Z}_e \rangle &= \frac{4}{3}\ell^3\alpha_3\varepsilon_{abcde}, \\ \langle \boldsymbol{J}_{ab}\boldsymbol{Z}_{cd}\boldsymbol{P}_e \rangle &= \frac{4}{3}\ell^3\alpha_3\varepsilon_{abcde}, \end{aligned}$$

Einstein-Hilbert action from 5D Chern-Simons

Chern-Simons theory is characterized by the action

$$S_{CS}^{(5D)} = \kappa \int d^5x \, \left(\frac{1}{\ell} R^{ab} R^{cd} e^e + \frac{2}{3} \frac{1}{\ell^3} R^{ab} e^c e^d e^e + \frac{1}{5} \frac{1}{\ell^5} e^a e^b e^c e^d e^e \right) \epsilon_{abcde}$$

$$L_{\mathrm{CS}}^{\mathfrak{B}_5} = \boxed{\alpha_1} \ell^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e + \boxed{\alpha_3} \varepsilon_{abcde} \left(\frac{2}{3} R^{ab} e^c e^d e^e + 2\ell^2 k^{ab} R^{cd} T^e + \ell^2 R^{ab} R^{cd} h^e \right)$$

when α_1 vanishes, and $T^a = 0$ is imposed, and solution is without matter $(k^{ab} = 0,\ h^a = 0)$

$$\delta L_{\mathrm{CS}}^{(5)} = 2\alpha_3 \varepsilon_{abcde} R^{ab} e^c e^d \delta e^e + \alpha_3 \ell^2 \varepsilon_{abcde} R^{ab} R^{cd} \delta h^e$$

$$\begin{cases} \varepsilon_{abcde} R^{ab} e^c e^d = 0 \\ \ell^2 \varepsilon_{abcde} R^{ab} R^{cd} = 0 \end{cases}$$

This action in the critic limit $\ell = 0$ leads to GR.

Pure Lovelock from 5D Chern -Simons

Pure Lovelock action

$$S=\int \alpha_1\epsilon_{abcde}\left(\frac{1}{l}R^{ab}R^{cd}e^e+\frac{1}{5l^5}e^ae^be^ce^de^e\right)$$

$$\epsilon_{abcde}\left(R^{ab}R^{cd}+\frac{1}{l^4}e^ae^be^ce^d\right)=0 \qquad \delta e^a$$

$$\epsilon_{abcde}R^{cd}T^e=0 \qquad \delta \omega^{ab}$$

$$\begin{aligned}S=\int &\alpha_1\epsilon_{abcde}\left(\frac{1}{l}R^{ab}R^{cd}e^e+\frac{1}{5l^5}e^ae^be^ce^de^e+\frac{2}{l^5}h^ah^be^ce^de^e\right.\\&+\frac{1}{l^5}h^ah^bh^ch^de^e+\frac{2}{l}Dk^{ab}R^{cd}h^e+\frac{1}{l}Dk^{ab}Dk^{cd}e^e+\frac{2}{3l^3}R^{ab}h^ch^dh^e\\&\left.+\frac{2}{3l^3}Dk^{ab}e^ce^de^e+\frac{2}{l^3}R^{ab}h^ce^de^e+\frac{2}{l^3}Dk^{ab}h^ch^de^e\right)\\&+\alpha_3\epsilon_{abcde}\left(\frac{2}{3l^3}R^{ab}e^ce^de^e+\frac{1}{l^5}h^ae^be^ce^de^e+\frac{2}{l^5}h^ah^bh^ce^de^e\right.\\&+\frac{1}{5l^5}h^ah^bh^ch^dh^e+\frac{2}{l^3}Dk^{ab}h^ce^de^e+\frac{2}{3l^3}Dk^{ab}h^ch^dh^e\\&\left.+\frac{2}{l^3}R^{ab}h^ch^de^e+\frac{1}{l}R^{ab}R^{cd}h^e+\frac{2}{l}Dk^{ab}R^{cd}e^e+\frac{1}{l}Dk^{ab}Dk^{cd}h^e\right)\end{aligned}$$

When the constant α_3 vanishes and a solution without matter $(k^{ab}=h^a=0)$ is considered,

$$\epsilon_{abcde}\left(R^{ab}R^{cd}+\frac{1}{l^4}e^ae^be^ce^d\right)=0, \qquad \delta e^a$$

$$\epsilon_{abcde}R^{ab}e^ce^d=0, \qquad \delta h^a$$

$$\epsilon_{abcde}R^{cd}T^e=0, \qquad \delta \omega^{ab}$$

$$\epsilon_{abcde}e^ce^dT^e=0. \qquad \delta k^{ab}$$

Interesting overconstraining of the solutions

Semigroup expansion and two algebraic families

Generators	Type $\mathfrak{B}_m$	Type $\mathfrak{C}_m$
J_{ab}, P_a	$\mathfrak{B}_3$ =Poincaré	$\mathfrak{C}_3$ =AdS
J_{ab}, P_a, Z_{ab}	$\mathfrak{B}_4$ =Maxwell	$\mathfrak{C}_4$ =AdS$\oplus$Lorentz
J_{ab}, P_a, Z_{ab}, R_a	$\mathfrak{B}_5$	$\mathfrak{C}_5$
$J_{ab}, P_a, Z_{ab}, R_a, \tilde{Z}_{ab}$	$\mathfrak{B}_6$	$\mathfrak{C}_6$
$\dots$	$\dots$	$\dots$

$$S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1} \qquad S_{\mathcal{M}}^{(N)} = \{\lambda_\alpha, \alpha = 0, \dots, N\}$$

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta} & \text{if } \alpha+\beta \leqslant N+1 \\ \lambda_{N+1} & \text{if } \alpha+\beta > N+1 \end{cases} \qquad \lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta} & \text{if } \alpha+\beta \leqslant N, \\ \lambda_{\alpha+\beta-2\left[\frac{N+1}{2}\right]} & \text{if } \alpha+\beta > N, \end{cases}$$

Automatic building blocks for the actions

- Connection Curvature forms Invariant tensors

- Born-Infeld theory in even dimensions
- Chern-Simons theory in odd dimensions

$$\mathcal{L}_{CS}^{2n+1}[A] = \kappa(n+1)\int_0^1 \delta t \left\langle A\left(tdA + t^2A^2\right)^n \right\rangle$$

AdS

$$\langle J_{a_1a_2} \cdots J_{a_{2n-1}a_{2n}} P_{a_{2n+1}} \rangle = \frac{2^n}{n+1}\epsilon_{a_1a_2\cdots a_{2n+1}}$$

New type of algebras

$$[P_a, P_b] = Z_{ab}\,,$$
$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b\,,$$
$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\,,$$
$$[J_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}\,,$$
$$[Z_{ab}, P_c] = \eta_{bc}R_a - \eta_{ac}R_b\,,$$
$$[J_{ab}, R_c] = \eta_{bc}R_a - \eta_{ac}R_b\,.$$

Generators

J_{ab}, P_a, Z_{ab}, R_a,

Algebra $\mathfrak{B}_5$

$$[R_a, R_b] = 0\,,$$
$$[Z_{ab}, R_c] = 0\,,$$
$$[Z_{ab}, Z_{cd}] = 0\,,$$
$$[R_a, P_b] = 0\,.$$

Inönü-Wigner contraction

$P_a \to \mu P_a\,,\qquad Z_{ab} \to \mu^2 Z_{ab}\,,\qquad$ and $\qquad R_a \to \mu^3 R_a$

Algebra $\mathfrak{C}_5$

$$[R_a, R_b] = Z_{ab}\,,$$
$$[Z_{ab}, R_c] = \eta_{bc}P_a - \eta_{ac}P_b\,,$$
$$[Z_{ab}, Z_{cd}] = \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\,,$$
$$[R_a, P_b] = J_{ab}\,,$$

Direct Maxwell algebra

$$\mathfrak{D}_5 = \text{AdS} \oplus \text{Poincaré}$$

$$[R_a, R_b] = Z_{ab}\,,$$
$$[Z_{ab}, R_c] = \eta_{bc}R_a - \eta_{ac}R_b\,,$$
$$[Z_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}\,,$$
$$[R_a, P_b] = Z_{ab}\,.$$

P. K. Concha, R. Durka, N. Merino and E. K. Rodrguez, *New family of Maxwell like algebras*, Phys. Lett. B **759**, 507 (2016) doi:10.1016/j.physletb.2016.06.016 [arXiv:1601.06443 [hep-th]].

Direct Maxwell algebra

$$\mathfrak{D}_m = AdS \oplus \mathfrak{B}_{m-2}$$

Inönü-Wigner contraction

$$P_a \to \mu P_a \,, \qquad Z_{ab} \to \mu^2 Z_{ab} \,, \qquad \text{and} \qquad R_a \to \mu^3 R_a \qquad \ldots$$

$$\mathfrak{D}_m \to \boxed{\mathfrak{B}_m} \leftarrow \mathfrak{C}_m$$ (the same limit)

Maxwell-like familes

m	Generators	Type $\mathfrak{B}_m$	Type $\mathfrak{C}_m$	Type $\mathfrak{D}_m$
3	J_{ab}, P_a	$\mathfrak{B}_3$ =Poincaré	$\mathfrak{C}_3$ =AdS	-
4	J_{ab}, P_a, Z_{ab}	$\mathfrak{B}_4$ =Maxwell	$\mathfrak{C}_4$ = AdS⊕Lorentz	-
5	J_{ab}, P_a, Z_{ab}, R_a	$\mathfrak{B}_5$	$\mathfrak{C}_5$	$\mathfrak{D}_5$ =AdS⊕Poincaré
6	$J_{ab}, P_a, Z_{ab}, R_a, \tilde{Z}_{ab}$	$\mathfrak{B}_6$	$\mathfrak{C}_6$	$\mathfrak{D}_6$ =AdS⊕Maxwell
...	...	...	...	...
m	$J_{ab}, P_a, Z_{ab}^{(i)}, R_a^{(j)}$	$\mathfrak{B}_m$	$\mathfrak{C}_m$	$\mathfrak{D}_m$ =AdS⊕$\mathfrak{B}_{m-2}$

What is thr freedom in closing the semigroup multiplication tables?

$\mathcal{B}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_0	λ_3	λ_2
λ_2	λ_2	λ_3	0_S	0_S
λ_3	λ_3	λ_2	0_S	0_S

$\mathcal{C}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_0	λ_3	λ_2
λ_2	λ_2	λ_3	λ_0	λ_1
λ_3	λ_3	λ_2	λ_1	λ_0

$\mathcal{D}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_0	λ_3	λ_2
λ_2	λ_2	λ_3	λ_2	λ_3
λ_3	λ_3	λ_2	λ_3	λ_2

$\mathfrak{B}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	0_S
λ_2	λ_2	λ_3	0_S	0_S
λ_3	λ_3	0_S	0_S	0_S

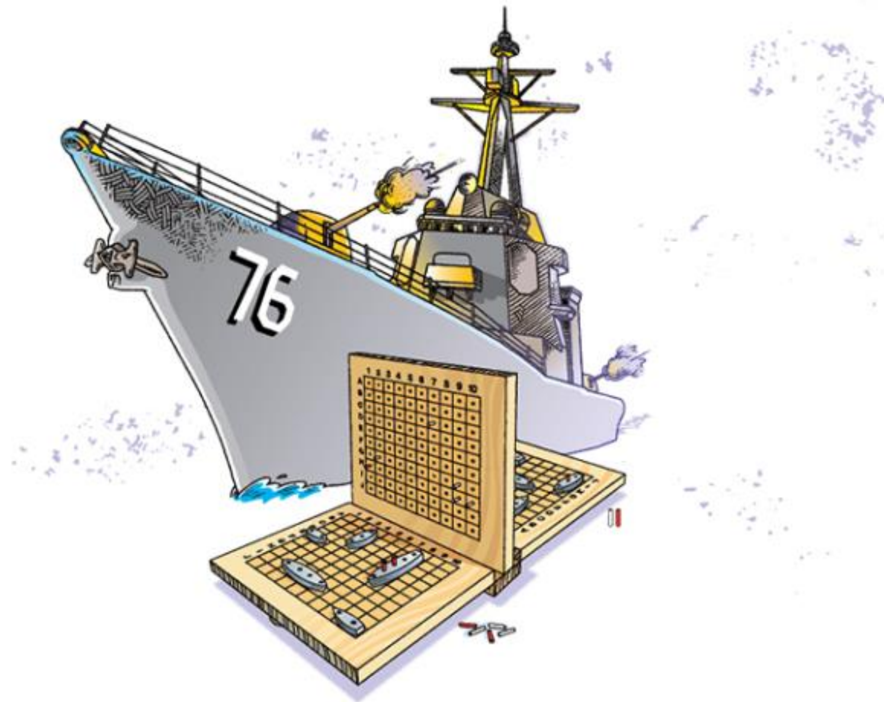
$\mathfrak{C}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	λ_0
λ_2	λ_2	λ_3	λ_0	λ_1
λ_3	λ_3	λ_0	λ_1	λ_2

$\mathfrak{D}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	λ_2
λ_2	λ_2	λ_3	λ_2	λ_3
λ_3	λ_3	λ_2	λ_3	λ_2

	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_1	0_S
λ_2	λ_2	λ_1	λ_2	0_S
λ_3	λ_3	0_S	0_S	0_S

	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_1	λ_2
λ_2	λ_2	λ_1	λ_2	λ_1
λ_3	λ_3	λ_2	λ_1	λ_2

	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_1	λ_2
λ_2	λ_2	λ_1	λ_2	λ_1
λ_3	λ_3	λ_2	λ_1	λ_0



$$\left[\tilde{J}_{ab}, \tilde{J}_{cd}\right] = \eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} - \eta_{bd}\tilde{J}_{ac} + \eta_{ad}\tilde{J}_{bc} \quad \rightarrow \quad R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b\,,$$

$$\left[\tilde{J}_{ab}, \tilde{P}_c\right] = \eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b \quad \rightarrow \quad T^a = de^a + \omega^a{}_b \wedge e^b\,.$$

Lorentz $J_{ab} = J_{ab,(0)} = \lambda_0 \tilde{J}_{ab}$

translation $P_a = P_{a,(0)} = \lambda_1 \tilde{P}_a$

$[J_{ab}, J_{cd}]$ and $[J_{ab}, P_c]$

$\lambda_0\lambda_0 = \lambda_0 \qquad \lambda_0\lambda_1 = \lambda_1$

Semigroup expansion

In the S-expansion method [2, 3] new algebras are derived from the AdS by using a particular choice of the abelian semigroup. The procedure starts from the decomposition of the original algebra $\mathfrak{g}$ into subspaces,

$$\mathfrak{g} = \mathfrak{so}\left(D-1,2\right) = \mathfrak{so}\left(D-1,1\right) \oplus \frac{\mathfrak{so}\left(D-1,2\right)}{\mathfrak{so}\left(D-1,1\right)} = V_0 \oplus V_1 ,$$

V_0 is spanned by the Lorentz generator $\tilde{J}_{ab}$
V_1 by the AdS translation generator $\tilde{P}_a$

The subspace structure $\quad [V_0, V_0] \subset V_0 , \quad [V_0, V_1] \subset V_1 , \quad [V_1, V_1] \subset V_0$

If we define $S = \{\lambda_0, \lambda_1, \ldots\}$ as an abelian semigroup with the multiplication law being

- associative: (ab)c=a(bc)
- commutative: ab=ba

then the Lie algebra $\mathfrak{G} = S \times \mathfrak{g}$ is called S-expanded algebra of $\mathfrak{g}$

F. Izaurieta, E. Rodríguez and P. Salgado, "Expanding Lie (super)algebras through Abelian semigroups," J. Math. Phys. **47**, 123512 (2006) [hep-th/0606215].

P. Salgado and S. Salgado, "$\mathfrak{so}(D-1,1) \otimes \mathfrak{so}(D-1,2)$ algebras and gravity," Phys. Lett. B **728**, 5 (2014).

J. Diaz, O. Fierro, F. Izaurieta, N. Merino, E. Rodríguez, P. Salgado and O. Valdivia, "A generalized action for $(2+1)$-dimensional Chern-Simons gravity," J. Phys. A **45**, 255207 (2012) [arXiv:1311.2215 [gr-qc]].

Semigroup expansion

If we define $S = \{\lambda_0, \lambda_1, \ldots\}$ as an abelian semigroup with the multiplication law being

- associative: (ab)c=a(bc)
- commutative: ab=ba

then the Lie algebra $\mathfrak{G} = S \times \mathfrak{g}$ is called S-expanded algebra of $\mathfrak{g}$

Resonant decomposition

$S_0 = \{\lambda_{2i}\}$ and $S_1 = \{\lambda_{2j+1}\}$ for $i, j = 0, 1, 2, \ldots$

resonant subset decomposition $S = S_0 \cup S_1$

This decomposition satisfies

$$S_0 \cdot S_0 \subset S_0\,, \quad S_0 \cdot S_1 \subset S_1\,, \quad S_1 \cdot S_1 \subset S_0$$
$$[V_0, V_0] \subset V_0\,, \quad [V_0, V_1] \subset V_1\,, \quad [V_1, V_1] \subset V_0$$

Generators of new algebra

new algebra will be spanned by the $\{J_{ab,(i)}, P_{a,(j)}\}$, where the new generators are related to original $\mathfrak{so}(D-1,2)$ ones through

$$J_{ab,(i)} = \lambda_{2i}\tilde{J}_{ab} \quad \text{and} \quad P_{a,(j)} = \lambda_{2j+1}\tilde{P}_a.$$

Invariant tensors

$$\left\langle (\lambda_{2k_1} \cdots \lambda_{2k_n}\lambda_{2k_{n+1}+1})\tilde{J}_{a_1a_2} \cdots \tilde{J}_{a_{2n-1}a_{2n}}\tilde{P}_{a_{2n+1}} \right\rangle = \sigma_\alpha \frac{2^n}{n+1}\epsilon_{a_1a_2\ldots a_{2n+1}}$$

$$\lambda_{2k_1} \cdots \lambda_{2k_n}\lambda_{2k_{n+1}+1} = \lambda_\alpha$$

arbitrary dimensionless constant σ_α, depending on the algebra, through its index introduces specific components of the invariant tensor

Multiplication table

$\bar{S}_0$ and $\bar{S}_1$ describing, respectively, elements later associated with the Lorentz-like and translation-like generators.

There is also possibility of absorbing element, for which $0_S\lambda_k = \lambda_k 0_S = 0_S$, ⟶ $0_S\mathbb{T}_M = 0$

$[J_{ab}, J_{cd}]$ and $[J_{ab}, P_c]$

$$\lambda_0\lambda_0 = \lambda_0$$

$$\lambda_0\lambda_1 = \lambda_1$$

	λ_0	λ_1
λ_0	λ_0	λ_1
λ_1	λ_1	$\lambda_1\lambda_1$

ISO	λ_0	λ_1
λ_0	λ_0	λ_1
λ_1	λ_1	0_S

AdS	λ_0	λ_1
λ_0	λ_0	λ_1
λ_1	λ_1	λ_0

$$[P_a, P_b] = \lambda_1\lambda_1[\tilde{P}_a, \tilde{P}_b] = \lambda_1\lambda_1\tilde{J}_{ab}$$

$$\lambda_1\lambda_1 \in \bar{S}_0 = \{0_S, \lambda_0\}$$

$[P_a, P_b] = 0$ and $[P_a, P_b] = J_{ab}$

That is not the only choice!

$\lambda_1\lambda_1$ multiplication could be closed by another element $\lambda_2 \in S_0$

	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	
λ_2	λ_2		

	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	$\lambda_1\lambda_2$
λ_2	λ_2	$\lambda_2\lambda_1$	$\lambda_2\lambda_2$

$$\lambda_0\lambda_2 = \lambda_0(\lambda_1\lambda_1) = (\lambda_0\lambda_1)\lambda_1 = \lambda_2$$

multiplication of $\lambda_1\lambda_2 = \lambda_2\lambda_1 = \lambda_1^3$

Maxwell-like algebras

	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	λ_1^3
λ_2	λ_2	λ_1^3	λ_1^4

$\mathfrak{B}_4$	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	0_S
λ_2	λ_2	0_S	0_S

$\mathfrak{C}_4$	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	λ_1
λ_2	λ_2	λ_1	λ_2

$\lambda_1^3 = 0_S$ and $\lambda_1^3 = \lambda_1$

$$
\begin{aligned}
[J_{ab}, J_{cd}] &= \lambda_0\lambda_0(\eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} + \eta_{ad}\tilde{J}_{bc} - \eta_{bd}\tilde{J}_{ac}) = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} + \eta_{ad}J_{bc} - \eta_{bd}J_{ac}\,,\\
[J_{ab}, Z_{cd}] &= \lambda_0\lambda_2(\eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} + \eta_{ad}\tilde{J}_{bc} - \eta_{bd}\tilde{J}_{ac}) = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} + \eta_{ad}Z_{bc} - \eta_{bd}Z_{ac}\,,\\
[Z_{ab}, Z_{cd}] &= \lambda_2\lambda_2(\eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} + \eta_{ad}\tilde{J}_{bc} - \eta_{bd}\tilde{J}_{ac}) = 0\,,\\
[J_{ab}, P_c] &= \lambda_0\lambda_1(\eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b) = \eta_{bc}P_a - \eta_{ac}P_b\,,\\
[Z_{ab}, P_c] &= \lambda_2\lambda_1(\eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b) = 0\,,\\
[P_a, P_b] &= \lambda_1\lambda_1\tilde{J}_{ab} = Z_{ab}\,.
\end{aligned}
$$

Lorentz $J_{ab} = J_{ab,(0)} = \lambda_0\tilde{J}_{ab}$

translation $P_a = P_{a,(0)} = \lambda_1\tilde{P}_a$

new generator $Z_{ab} = J_{ab,(1)} = \lambda_2\tilde{J}_{ab}$

$\mathfrak{B}_4$	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	0_S
λ_2	λ_2	0_S	0_S

semigroup $\leftrightarrow$ *algebra*

$[X_{..}, X_{..}]$, $[X_{..}, X_{.}]$ and $[X_{.}, X_{.}]$

$0_S \mathbb{T}_M = 0$

[,]	$J_{..}$	$P_{.}$	$Z_{..}$
$J_{..}$	$J_{..}$	$P_{.}$	$Z_{..}$
$P_{.}$	$P_{.}$	$Z_{..}$	0
$Z_{..}$	$Z_{..}$	0	0

Maxwell-like algebras

new element $\lambda_3 \in S_1$ → new translational-like generator $R_a = P_{a,(1)} = \lambda_3 \tilde{P}_a$,

	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	$\lambda_1\lambda_3$
λ_2	λ_2	λ_3	$\lambda_2\lambda_2$	$\lambda_2\lambda_3$
λ_3	λ_3	$\lambda_3\lambda_1$	$\lambda_3\lambda_2$	$\lambda_3\lambda_3$

$\lambda_1\lambda_3 \in \{0_S, \lambda_0, \lambda_2\}$

$\mathfrak{B}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	0_S
λ_2	λ_2	λ_3	0_S	0_S
λ_3	λ_3	0_S	0_S	0_S

all zeros,

$\mathfrak{C}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	λ_0
λ_2	λ_2	λ_3	λ_0	λ_1
λ_3	λ_3	λ_0	λ_1	λ_2

anti-diagonal pattern,

$\mathfrak{D}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	λ_2
λ_2	λ_2	λ_3	λ_2	λ_3
λ_3	λ_3	λ_2	λ_3	λ_2

chessboard pattern

Algebra $\mathfrak{B}_5$: relation between **Chern-Simons (CS) gravity** and **Einstein-Hilbert action** (later the same in even dimensions for Born-Infeld (BI) gravity and GR)

J. D. Edelstein, M. Hassaine, R. Troncoso and J. Zanelli, "Lie-algebra expansions, Chern-Simons theories and the Einstein-Hilbert Lagrangian," Phys. Lett. B **640**, 278 (2006) [hep-th/0605174].

F. Izaurieta, E. Rodríguez, P. Minning, P. Salgado and A. Perez, "Standard General Relativity from Chern-Simons Gravity," Phys. Lett. B **678**, 213 (2009) [arXiv:0905.2187 [hep-th]].

Algebra $\mathfrak{C}_5$ was discussed in the context of so called **Pure Lovelock (PL) action**, which instead of the full Lanczos-Lovelock series **contains only the cosmological constant term** and **single *p* power polynomial term in the Riemann curvature** (in 5D it can be either RRe or Re^3).

P. K. Concha, R. Durka, C. Inostroza, N. Merino and E. K. Rodríguez, "Pure Lovelock gravity and Chern-Simons theory," arXiv:1603.09424 [hep-th].

Algebra $\mathfrak{D}_5$ admits direct sum of the AdS ⨁ Poincaré which effectively makes the **CS/BI action** to be expressed as a **sum of two independent pieces**.

P.K. Concha, R. Durka, N. Merino and E.K. Rodríguez, *New family of Maxwell like algebras*, Phys. Lett. B **759** (2016) 507, arXiv:1601.06443 [hep-th].

Towards generalization

Algebras for $m=7$ with 6 generators

$\mathfrak{B}_7$	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
λ_0	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
λ_1	λ_1	λ_2	λ_3	λ_4	λ_5	0_S
λ_2	λ_2	λ_3	λ_4	λ_5	0_S	0_S
λ_3	λ_3	λ_4	λ_5	0_S	0_S	0_S
λ_4	λ_4	λ_5	0_S	0_S	0_S	0_S
λ_5	λ_5	0_S	0_S	0_S	0_S	0_S

$\mathfrak{C}_7$	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
λ_0	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
λ_1	λ_1	λ_2	λ_3	λ_4	λ_5	λ_0
λ_2	λ_2	λ_3	λ_4	λ_5	λ_0	λ_1
λ_3	λ_3	λ_4	λ_5	λ_0	λ_1	λ_2
λ_4	λ_4	λ_5	λ_0	λ_1	λ_2	λ_3
λ_5	λ_5	λ_0	λ_1	λ_2	λ_3	λ_4

Total number of possible patterns for arbitrary (m-1) types f generators is equal

$$\left[\frac{m+1}{2}\right]$$

$\mathfrak{D}_7$	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
λ_0	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
λ_1	λ_1	λ_2	λ_3	λ_4	λ_5	λ_4
λ_2	λ_2	λ_3	λ_4	λ_5	λ_4	λ_5
λ_3	λ_3	λ_4	λ_5	λ_4	λ_5	λ_4
λ_4	λ_4	λ_5	λ_4	λ_5	λ_4	λ_5
λ_5	λ_5	λ_4	λ_5	λ_4	λ_5	λ_4

$\mathfrak{E}_7$	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
λ_0	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
λ_1	λ_1	λ_2	λ_3	λ_4	λ_5	λ_2
λ_2	λ_2	λ_3	λ_4	λ_5	λ_2	λ_3
λ_3	λ_3	λ_4	λ_5	λ_2	λ_3	λ_4
λ_4	λ_4	λ_5	λ_2	λ_3	λ_4	λ_5
λ_5	λ_5	λ_2	λ_3	λ_4	λ_5	λ_2

General law (excluding B_m family)

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}\,, & \text{for} \quad \alpha+\beta \leq m-2 \\ \lambda_\gamma\,, & \text{for} \quad \alpha+\beta > m-2 \end{cases}$$

$$\gamma = (\alpha+\beta-(m-1)) \bmod ((m-1)-\rho)+\rho$$

$\mathfrak{B}_m$ family is retrieved as Inönü-Wigner contraction of all these algebras in the limit of dimensionless parameter $\mu \to \infty$ scaling generators $P_{a,(0)} \to \mu P_{a,(0)}\,, J_{ab,(1)} \to \mu^2 J_{ab,(1)}\,, P_{a,(1)} \to \mu^3 P_{a,(1)}, \ldots$

R. Durka, *Resonant algebras and gravity*, arXiv:1605.00059 [hep-th].

Poincaré-like, AdS-like, Maxwell-like algebras

	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	0_S	
λ_2	λ_2		

	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_0	
λ_2	λ_2		

	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	
λ_2	λ_2		

Generators

$J_{ab},\ P_a,\ Z_{ab}$

- $4\times$ Poincaré-like algebras we could denote as type B_4, BC_4, CB_4, and $C_4 \equiv ISO \oplus Lorentz$:

B_4	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	0_S	0_S
λ_2	λ_2	0_S	0_S

BC_4	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	0_S	0_S
λ_2	λ_2	0_S	λ_2

CB_4	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	0_S	λ_1
λ_2	λ_2	λ_1	0_S

C_4	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	0_S	λ_1
λ_2	λ_2	λ_1	λ_2

- none of the AdS-like algebra (associativity is not fulfilled in any configuration)

- $2\times$ Maxwell-like algebras of type $\mathfrak{B}_4$ and $\mathfrak{C}_4$ already introduced in a previous section:

$\mathfrak{B}_4$	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	0_S
λ_2	λ_2	0_S	0_S

$\mathfrak{C}_4$	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	λ_1
λ_2	λ_2	λ_1	λ_2

Generators

$\{J_{ab}, P_a, Z_{ab}, R_a\}$

17 x Poincaré-like,
3 x AdS-like,
10 x Maxwell-like

$\mathcal{C}_5$, corresponds to the Klein group

$\mathfrak{C}_5$, correspond to the cyclic group $\mathbb{Z}_4$

- explicit tables for the algebras labeled by m = 3, 4, 5, 6,
- tool checking semigroup associativity

resonantalgebras.wordpress.com

Further enlargement, coming with the new translational $R_a = P_{a,(1)} = \lambda_3 \tilde{P}_a$ generator, brings much richer collection of the algebras:

- $17\times$ Poincaré-like
- $3\times$ AdS-like of type $\mathcal{B}_5$, $\mathcal{C}_5$, and $\mathcal{D}_5 \equiv AdS \oplus AdS$

$\mathcal{B}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_0	λ_3	λ_2
λ_2	λ_2	λ_3	0_S	0_S
λ_3	λ_3	λ_2	0_S	0_S

$\mathcal{C}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_0	λ_3	λ_2
λ_2	λ_2	λ_3	λ_0	λ_1
λ_3	λ_3	λ_2	λ_1	λ_0

$\mathcal{D}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_0	λ_3	λ_2
λ_2	λ_2	λ_3	λ_2	λ_3
λ_3	λ_3	λ_2	λ_3	λ_2

- $10\times$ Maxwell-like: three of them $\mathfrak{B}_5, \mathfrak{C}_5, \mathfrak{D}_5$ were already derived in a previous section

$\mathfrak{B}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	0_S
λ_2	λ_2	λ_3	0_S	0_S
λ_3	λ_3	0_S	0_S	0_S

$\mathfrak{C}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	λ_0
λ_2	λ_2	λ_3	λ_0	λ_1
λ_3	λ_3	λ_0	λ_1	λ_2

$\mathfrak{D}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	λ_2
λ_2	λ_2	λ_3	λ_2	λ_3
λ_3	λ_3	λ_2	λ_3	λ_2

but surprisingly there are five others with the 0_S (from them only one is being presented below) and additional two without the zero elements

	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_1	0_S
λ_2	λ_2	λ_1	λ_2	0_S
λ_3	λ_3	0_S	0_S	0_S

	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_1	λ_2
λ_2	λ_2	λ_1	λ_2	λ_1
λ_3	λ_3	λ_2	λ_1	λ_2

	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_1	λ_2
λ_2	λ_2	λ_1	λ_2	λ_1
λ_3	λ_3	λ_2	λ_1	λ_0

Gravity actions

Algebra-valued one-form connection

$$A = \frac{1}{2}\omega^{ab,(i)} J_{ab,(i)} + \frac{1}{\ell} e^{a,(j)} P_{a,(j)}$$

Gauge parameter

$$\Theta = \frac{1}{2}\Lambda^{ab,(i)} J_{ab,(i)} + \xi^{a,(j)} P_{a,(j)}$$

Infinitesimal gauge transformation

$$\begin{aligned}
\delta_\Theta \omega^{ab,(i)} J_{ab,(i)} &= -\left(d\Lambda^{ab,(i)} J_{ab,(i)} + (\omega^{ac,(i)} \Lambda_c{}^{b,(p)} + \omega^{bc,(i)} \Lambda_c^{a,(p)}) \lambda_{2i}\lambda_{2p} \tilde{J}_{ab} \right) \\
&\quad - \frac{1}{\ell}(e^{a,(j)}\xi^{b,(q)} - e^{b,(j)}\xi^{a,(q)}) \lambda_{2j+1}\lambda_{2q+1} \tilde{J}_{ab} \,, \\
\delta_\Theta \frac{1}{\ell} e^{a,(j)} P_{a,(j)} &= -\left(d\xi^{a,(j)} P_{a,(j)} + \omega^{a}{}_{b}{}^{,(i)} \xi^{b,(j)} \lambda_{2i}\lambda_{2j+1} \tilde{P}_a \right) \\
&\quad + \Lambda^{a}{}_{b}{}^{,(i)} \frac{1}{\ell} e^{b,(j)} \lambda_{2i}\lambda_{2j+1} \tilde{P}_a \,.
\end{aligned}$$

$$\delta_\Theta \omega_\mu^{ab} = -D_\mu^\omega \Lambda^{ab} + \ldots$$

$$\frac{1}{\ell}\delta_\Theta e_\mu^a = -D_\mu^\omega \xi^a + \frac{1}{\ell}\Lambda^a{}_b\, e_\mu^b + \ldots$$

Curvature two-form

$$\begin{aligned}
F &= \frac{1}{2}\left(d\omega^{ab,(i)}\lambda_{2i} + \omega^{ac,(i)}\omega_c{}^{b,(p)}\lambda_{2i}\lambda_{2p} + \frac{1}{\ell^2} e^{a,(j)} e^{b,(q)} \lambda_{2j+1}\lambda_{2q+1} \right) \tilde{J}_{ab} \\
&\quad + \frac{1}{\ell}\left(de^{a,(j)}\lambda_{2j+1} + \omega^{a}{}_{b}{}^{,(i)} e^{b,(j)} \lambda_{2i}\lambda_{2j+1} \right) \tilde{P}_a \,.
\end{aligned}$$

Invariant tensors

AdS $\langle J_{a_1a_2}\cdots J_{a_{2n-1}a_{2n}}P_{a_{2n+1}}\rangle = \frac{2^n}{n+1}\epsilon_{a_1a_2\cdots a_{2n+1}}$

Chern-Simons in odd dimensions

$$\langle J_{a_1a_2,(k_1)}\cdots J_{a_{2n-1}a_{2n},(k_n)}P_{a_{2n+1},(k_{n+1})}\rangle = \sigma_{2j+1}\delta^j_{k(k_1,k_2,\ldots,k_{n+1})}\frac{2^n}{n+1}\epsilon_{a_1a_2\ldots a_{2n+1}}$$

arbitrary dimensionless constant σ_α, depending on the algebra,

through its index introduces specific components of the invariant tensor

$$\left\langle(\lambda_{2k_1}\cdots\lambda_{2k_n}\lambda_{2k_{n+1}+1})\tilde{J}_{a_1a_2}\cdots\tilde{J}_{a_{2n-1}a_{2n}}\tilde{P}_{a_{2n+1}}\right\rangle = \sigma_\alpha\frac{2^n}{n+1}\epsilon_{a_1a_2\ldots a_{2n+1}}$$

$$\lambda_{2k_1}\cdots\lambda_{2k_n}\lambda_{2k_{n+1}+1} = \lambda_\alpha$$

Born-Infeld in even dimensions

$$\left\langle(\lambda_{2k_1}\cdots\lambda_{2k_n})\tilde{J}_{a_1a_2}\cdots\tilde{J}_{a_{2n-1}a_{2n}}\right\rangle = \sigma_\alpha\frac{2^{n-1}}{n}\epsilon_{a_1a_2\ldots a_{2n}}$$

Gravity actions

Chern-Simons in odd dimensions

$5D$ CS	Poincaré-like	Maxwell		AdS-like	Maxwell-like				
terms	$m=4$ and 5	$\mathfrak{B}_4$	$\mathfrak{C}_4$	$m=5$	$\mathfrak{B}_5$	$\mathfrak{C}_5$	$\mathfrak{D}_5$	tables from (27)	rest with 0_s
RRe	σ_1	σ_1	σ_1	σ_1	σ_1	σ_1	σ_1	σ_1	σ_1
Re^3	0	0	σ_1	σ_1	σ_3	σ_3	σ_3	σ_1	0
e^5	0	0	σ_1	σ_1	0	σ_1	σ_3	σ_1	0

Born-Infeld in even dimensions

$4D$ BI	Poincaré-like	Maxwell		AdS-like	Maxwell-like				
terms	$m=4$ and 5	$\mathfrak{B}_4$	$\mathfrak{C}_4$	$m=5$	$\mathfrak{B}_5$	$\mathfrak{C}_5$	$\mathfrak{D}_5$	tables from (27)	rest with 0_s
RR	σ_0	σ_0	σ_0	σ_0	σ_0	σ_0	σ_0	σ_0	σ_0
Re^2	0	σ_2	σ_2	σ_0	σ_2	σ_2	σ_2	σ_2	σ_2
e^4	0	0	σ_2	σ_0	0	σ_0	σ_2	σ_2	0

- Imposing specific conditions on the wide class of the S-expanded algebras not only assured us consisted grounds for many valuable and interesting algebras, but it also limited overwhelming vastness of an algebraic examples.
- Expansions for the abelian semigroups with the zero and identity, simultaneously obeying the resonant decomposition, are well defined and straightforwardly applicable in the gravity context.
- Non-trivial extension to the construction of gauge gravity theories.
- Finding all known Maxwell families of diferent types and generalize their description to the new examples.
- Certainly, there is still a lot of work required to find satisfactory interpretation of the extended field content, description of the new symmetries and understand well their consequences…

Thanks!